# 12) Appendix I: Source-free vector field solutions

Vector wave equations in source-free regions

In a homogeneous isotropic medium away from sources, if $\vec{\mathcal{E}}$ denotes $\vec{\mathcal{E}}, \vec{\mathcal{D}}, \vec{\mathcal{H}}, \vec{\mathcal{B}}$ or $\vec{\mathcal{A}}$:

$$\vec{\nabla}^2 \vec{\mathcal{E}} - \mu\varepsilon \frac{\partial^2 \vec{\mathcal{E}}}{\partial t^2} = 0$$

with $\vec{\nabla}^2 \vec{\mathcal{E}} = \vec{\nabla}(\vec{\nabla}.\vec{\mathcal{E}}) - \vec{\nabla}\times\vec{\nabla}\times\vec{\mathcal{E}}$

Then, under $e^{j\omega t}$ time convention:

$$(1) \quad \boxed{\vec{\nabla}\vec{\nabla}.\vec{\mathcal{E}} - \vec{\nabla}\times\vec{\nabla}\times\vec{\mathcal{E}} + k^2\vec{\mathcal{E}} = \vec{0}} \quad , \; \underline{k^2 = \omega^2\mu\varepsilon}$$

__Proof:__

Maxwell's equations in homogeneous media, away from sources

$$\begin{cases} \vec{\nabla}\times\vec{\mathcal{E}} = -\dfrac{\partial \mathcal{B}}{\partial t} = -\mu\dfrac{\partial\vec{\mathcal{H}}}{\partial t} \\ \vec{\nabla}\times\vec{\mathcal{H}} = \dfrac{\partial \mathcal{D}}{\partial t} = \varepsilon\dfrac{\partial\vec{\mathcal{E}}}{\partial t} \end{cases} \qquad \begin{cases} \vec{\nabla}.\vec{\mathcal{E}} = 0 \\ \vec{\nabla}.\vec{\mathcal{B}} = 0 \end{cases}$$

$$\Rightarrow \vec{\nabla}\times\vec{\nabla}\times\vec{\mathcal{E}} + \mu_{\cancel{0}}\,\varepsilon_{\cancel{0}}\frac{\partial^2\vec{\mathcal{E}}}{\partial t^2} = 0 \Rightarrow \vec{\nabla}(\vec{\nabla}.\vec{\mathcal{E}}) - \vec{\nabla}\times\vec{\nabla}\times\vec{\mathcal{E}} - \mu_{\cancel{0}}\,\varepsilon_{\cancel{0}}\frac{\partial^2\vec{\mathcal{E}}}{\partial t^2} = \vec{0}$$

$$\Rightarrow \underline{\vec{\nabla}(\vec{\nabla}.\vec{\mathcal{E}}) - \vec{\nabla}\times\vec{\nabla}\times\vec{\mathcal{E}} + k^2\vec{\mathcal{E}} = 0} \quad , \ k^2 = \omega^2\mu\varepsilon$$

Same for $\vec{\mathcal{H}}$, $\vec{\mathcal{B}}$ or $\vec{\mathcal{D}}$.

$$\vec{\nabla}\times\vec{\mathcal{B}} = \vec{\nabla}\times\vec{\nabla}\times\vec{\mathcal{A}} = \mu\varepsilon\frac{\partial\vec{\mathcal{E}}}{\partial t} = j\omega\mu\varepsilon\left(-\vec{\nabla}\phi - j\omega\vec{\mathcal{A}}\right)$$

$$\Rightarrow \vec{\nabla}\times\vec{\nabla}\times\vec{\mathcal{A}} - k^2\vec{\mathcal{A}} = -j\omega\mu\varepsilon\vec{\nabla}\phi$$

Lorentz gauge: $\vec{\nabla}.\vec{A} = -j\omega\mu\varepsilon\,\phi$

$$\Rightarrow \underline{\vec{\nabla}(\vec{\nabla}.\vec{A}) - \vec{\nabla}\times\vec{\nabla}\times\vec{A} + k^2\vec{A} = 0} \qquad [QED]$$

Construction of independent vector solutions of (1)

Let $\psi$ be a solution of the scalar wave equation:

$$\vec{\nabla}^2\psi + k^2\psi = 0$$

and $\vec{a}$ any constant vector of unit length. Then:

$$\boxed{\vec{L} = \vec{\nabla}\psi \qquad \vec{M} = \vec{\nabla}\times\psi\vec{a} \qquad \vec{N} = \frac{1}{k}\vec{\nabla}\times\vec{M}} \quad (2)$$

are all solutions of (1)

Proof:

$\boxed{\vec{L}}$: $\quad \vec{\nabla}^2\vec{L} = \vec{\nabla}^2\vec{\nabla}\psi = \begin{bmatrix} (\partial_{x^2} + \partial_{y^2} + \partial_{z^2})\,\partial_x\psi \\ (\quad)\,\partial_y\psi \\ (\quad)\,\partial_z\psi \end{bmatrix} = \vec{\nabla}(\vec{\nabla}^2\psi)$

$$\Rightarrow \vec{\nabla}^2\vec{L} + k^2\vec{L} = \vec{\nabla}(\vec{\nabla}^2\psi + k^2\psi) = 0 \quad \text{QED}$$

$\boxed{\vec{M}}$: $\quad \vec{M} = \vec{\nabla}\times\psi\vec{a} = \vec{\nabla}\psi\times\vec{a} + \psi\underbrace{\vec{\nabla}\times\vec{a}}_{\vec{0}} = \vec{L}\times\vec{a}$

$$(\vec{\nabla}^2 L)\times\vec{a} = \begin{pmatrix} \nabla^2 L_x \\ \nabla^2 L_y \\ \nabla^2 L_z \end{pmatrix} \times \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} = \begin{pmatrix} (\nabla^2 L_y)a_z - (\nabla^2 L_z)a_y \\ (\nabla^2 L_z)a_x - (\nabla^2 L_x)a_z \\ (\nabla^2 L_x)a_y - (\nabla^2 L_y)a_x \end{pmatrix} = \nabla^2(\vec{L}\times\vec{a})$$

$$\Rightarrow \vec{\nabla}^2 M + k^2\vec{M} = (\vec{\nabla}^2\vec{L} + k^2\vec{L})\times\vec{a} = 0 \quad \text{QED}$$

$\boxed{\vec{N}}$: $\quad \vec{N} = \frac{1}{k}\vec{\nabla}\times\vec{M} \Rightarrow \vec{\nabla}^2\vec{N} = \frac{1}{k}\vec{\nabla}^2(\vec{\nabla}\times\vec{M}) = \frac{1}{k}\vec{\nabla}\times(\vec{\nabla}^2\vec{M})$

$$\Rightarrow \vec{\nabla}^2\vec{N} + k^2\vec{N} = \frac{1}{k}\vec{\nabla}\times(\vec{\nabla}^2\vec{M} + k^2\vec{M}) = 0 \quad \text{QED}$$

Properties :

* $\vec{M} = \vec{L} \times \vec{a} \Rightarrow \vec{M} \cdot \vec{L} = 0$

* $\vec{L}$ is irrotational : $\vec{\nabla} \times \vec{L} = 0 \qquad \vec{\nabla} \cdot \vec{L} = \nabla^2 \psi = -k^2 \psi$

* $\vec{M}$ and $\vec{N}$ are solenoidal : $\vec{\nabla} \cdot \vec{M} = 0 \quad , \quad \vec{\nabla} \cdot \vec{N} = 0$

* $\vec{M}$ and $\vec{N}$ are proportional to the curl of the other

$$\vec{N} = \frac{1}{k} \vec{\nabla} \times \vec{M}$$

$$\vec{\nabla} \times \vec{N} = \frac{1}{k} \vec{\nabla} \times \vec{\nabla} \times \vec{M} = \frac{1}{k} \vec{\nabla} \times \left( \vec{\nabla} \times (\vec{L} \times \vec{a}) \right)$$

$$= \frac{1}{k} \vec{\nabla} \times \left( \overbrace{(\vec{a} \cdot \vec{\nabla}) \vec{L}}^{\vec{0}} - \underbrace{\vec{a}\, \vec{\nabla} \cdot \vec{L}}_{k^2 \psi \vec{a}} \right) = k \vec{\nabla} \times (\psi \vec{a}) = k \vec{M}$$

$$\Rightarrow \vec{M} = \frac{1}{k} \vec{\nabla} \times \vec{N}$$

Electromagnetic field expansions

Suppose $\nabla^2 \psi + k^2 \psi$ has discrete solutions indexed by $n$, $\psi_n$

$\Rightarrow L_n, M_n, N_n$ generate the vector field solutions of (1)

$$\boxed{\begin{aligned}
&\Rightarrow \vec{A} = \frac{1}{j\omega} \sum_n \left(a_n \vec{M}_n + b_n \vec{N}_n + c_n \vec{L}_n\right) \\
&\Rightarrow \vec{E} = -\sum_n \left(a_n \vec{M}_n + b_n \vec{N}_n\right) \\
&\vec{H} = \frac{k}{j\omega\mu} \sum_n \left(a_n \vec{N}_n + b_n \vec{M}_n\right) \\
&\phi = -\sum_n c_n \psi_n
\end{aligned}}$$

Proof:

$$* \quad \vec{H} = \frac{1}{\mu} \vec{\nabla} \times \vec{A} = \frac{1}{j\mu\omega} \sum_n \Big\{ a_n \underbrace{\vec{\nabla} \times \vec{M}_n}_{k\vec{N}_n} + b_n \underbrace{\vec{\nabla} \times \vec{N}_n}_{k\vec{M}_n} + c_n \underbrace{\vec{\nabla} \times \vec{L}_n}_{\vec{0}} \Big\}$$

$$\Rightarrow \vec{H} = \frac{k}{j\omega\mu} \sum_n \left(a_n \vec{N}_n + b_n \vec{M}_n\right)$$

$$* \quad \vec{E} = \frac{1}{j\varepsilon\omega} \vec{\nabla} \times \vec{H} = -\cancelto{1}{\frac{k^2}{\omega^2\mu\varepsilon}} \sum_n \left(a_n \vec{M}_n + b_n \vec{N}_n\right)$$

$$* \quad \phi = \frac{1}{-j\omega\mu\varepsilon} \vec{\nabla} \cdot \vec{A} = -\frac{1}{j^2\omega^2\mu\varepsilon} \sum_n a_n \cancelto{0}{\vec{\nabla} \cdot \vec{M}_n} + b_n \cancelto{0}{\vec{\nabla} \cdot \vec{N}_n} + c_n \underbrace{\vec{\nabla} \cdot \vec{L}_n}_{\nabla^2 \psi_n = -k^2 \psi_n}$$

$$\phi = -\sum_n c_n \psi_n$$

Application to plane waves

We take $\psi = e^{-j\vec{k}\cdot\vec{r}}$ , $\vec{L} = \vec{\nabla}\psi = -j\vec{k}\psi$

$$\vec{M} = \vec{\nabla}\times \psi\vec{a} = \vec{L}\times\vec{a} = -j\psi\vec{k}\times\vec{a}$$

$$\vec{N} = \frac{1}{k}\vec{\nabla}\times\vec{M} = \frac{1}{k}\vec{\nabla}\times\left(-j\psi\underbrace{\vec{k}\times\vec{a}}_{\text{constant vector}}\right)$$

$$= -\frac{j}{k}\underbrace{\vec{\nabla}\psi}_{-j\vec{k}\psi}\times\vec{k}\times\vec{a} = -\frac{1}{k}\psi\,\vec{k}\times\vec{k}\times\vec{a}$$

$$\Rightarrow \boxed{\vec{L} = -j\vec{k}\psi \quad , \quad \vec{M} = -j\psi\,\vec{k}\times\vec{a} \quad , \quad \vec{N} = \frac{1}{k}\psi\,(\vec{k}\times\vec{a})\times\vec{k}}$$

plane waves

$$\begin{cases} \vec{E} = -\left(a\vec{M} + b\vec{N}\right) \\ \vec{H} = \dfrac{k}{j\omega\mu}\left(\underset{\substack{\uparrow \\ \text{TE}}}{a\vec{N}} + \underset{\substack{\uparrow \\ \text{TM}}}{b\vec{M}}\right) \end{cases}$$